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THE THREE-DIMENSIONAL POINCARÉ CONTINUED FRACTION ALGORITHM

BY

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ABSTRACT

It is proved here that for Lebesgue-almost every line in the three-dimensional Euclidean space, the Poincaré continued fraction algorithm fixes a vertex. Besides, the algorithm is nonergodic, although the Gauss map, defined by the algorithm, has an attractor and is ergodic. It is also shown that the Euclidean algorithm and the horocycle flow are orbit equivalent.

1. Introduction

The simultaneously rational approximation of points in $\mathbb{R}^n$, for $n \geq 2$, the so-called multidimensional continued fraction expansions, is the current topic of study. With this in mind, it is worthwhile recalling some ideas of Poincaré. The approximation method proposed by Poincaré deserves attention because it bears a simple geometrical interpretation. As will be seen, it can be extended to any other multidimensional continued fraction algorithm.

Next, we recall the Poincaré [Po1] geometrical interpretation of continued fractions.

Consider a parallelogram (or a rectangle or a square) and tile the plane with it. Choose a basic parallelogram OABC; this is defined by the dihedron OAC (Figure 1a), where O denotes the origin (Figure 1b).

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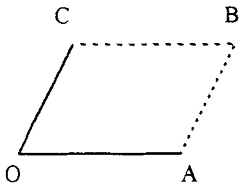


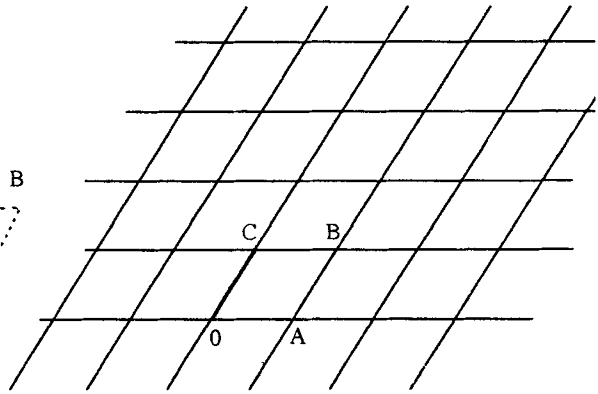
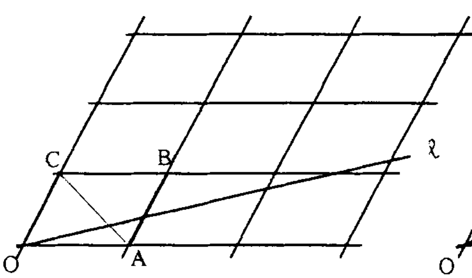


Figure 1a

Figure 1b

Let $\alpha > 0$ be the number to be approximated. Let ℓ be the line $y = \alpha x$. We assume ℓ is in the interior of the basic parallelogram. This line gets out of the parallelogram through the side AB or BC. (Figure 2a). Assume this side is AB. Let D be the symmetric point of O with respect to the middle point of AB. The parallelogram OADB (Figure 2b) enjoys the same properties as the parallelogram OABC. In this way, a sequence of parallelograms enjoying these properties is obtained. The vertices which are common to at least two parallelograms correspond to the approximants of α.



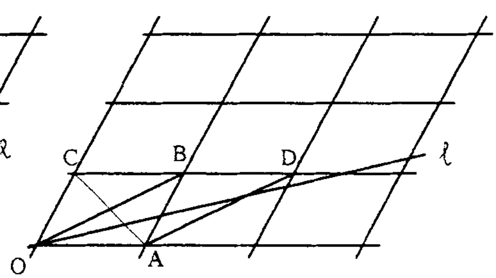


Figure 2a

Figure 2b

This procedure defines an algorithm, that to α is associated a sequence of plane tilings by parallograms or a chain of vertices from the simplex. It is convenient to introduce an analytical version of this algorithm. Let x_1, x_2 be the affine coordinates of the point where the line ℓ intersects the segment AC (Figure 2a) with respect to the basis OA, OC. Let OA$'$B$'$C$'$ be the next parallelogram and x_1', x_2' be the affine coordinates of the same point with respect to the new basis OA′, OC′. Analytically, this corresponds to the following algorithm F: if $0 \leq x_1 \leq x_2$

$$F(x_1, x_2) = F(x_2, x_1) = (x_1, x_2 - x_1). \tag{1.1}$$

This is the **Euclidean algorithm** defined step by step.

In 1884, Poincaré [Po2] generalized his idea in order to approximate two num bers $\alpha, \beta > 0$. Let O, A, B, C be the vertices of a tetrahedron (Figure 3a) in the space $\mathbb{R}^3$.

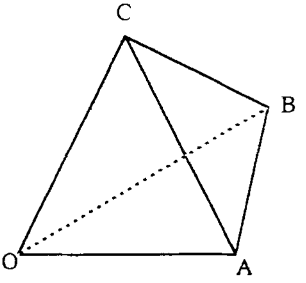


Figure 3a

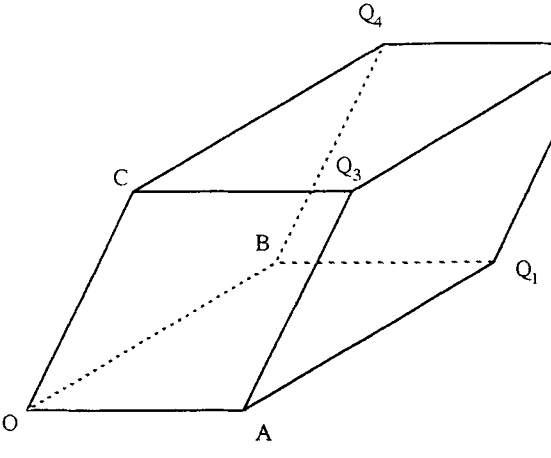


Figure 3b

Complete the tetrahedron OABC in order to obtain the parallelepiped $OABCQ_1Q_2Q_3Q_4$ (Figure 3b) and tile the space with it. Here the point O means the origin. Consider the line $\ell, y = \alpha x, z = \beta x$ and assume it intersects the interior of the parallelepiped. The parallelepiped is partitioned into six tetrahedrons: OAQ_1Q_2, OAQ_2Q_3, OBQ_1Q_2, OBQ_2Q_4, OCQ_2Q_3, OCQ_2Q_4. The six tetrahedrons defined by them have equal volumes. Consider that tetrahedron which intersects the line ℓ in its interior (Figure 4).

This new tetrahedron OA′B′C′ enjoys the same properties as the tetrahedron OABC. In this way, a sequence of tetrahedrons enjoying these properties is obtained. As before, this procedure defines an algorithm which associates to each pair of numbers α, β a sequence of tetrahedrons or a chain of vertices from the simplex:

$$\mathrm{A,B,C} \mapsto \mathrm{A}', \mathrm{B}', \mathrm{C}' \mapsto \cdots \mapsto \mathrm{A}^{(k)}, \mathrm{B}^{(k)}, \mathrm{C}^{(k)} \mapsto \cdots$$

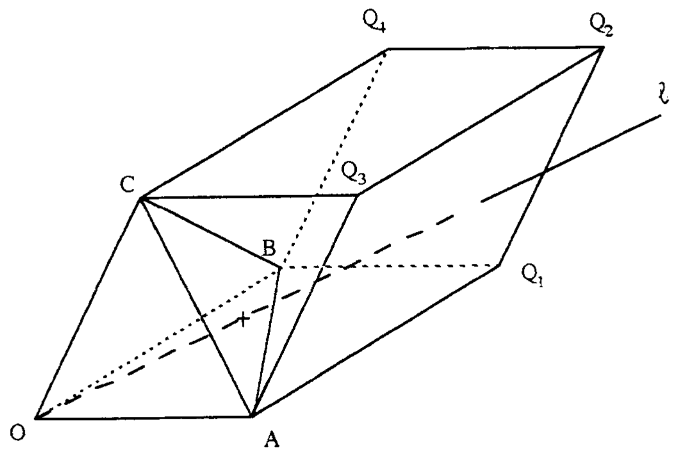


Figure 4

It will be proved that this algorithm, on the other hand, does not possess the same fundamental property enjoyed by the classical continued fraction algorithm on the plan as was expected by Poincaré. It will be seen here that an interesting phenomenon occurs: the two-dimensional continued fraction algorithm embedded in this three-dimensional algorithm prevails over it. The following assertion will be proved here:

(1.2) "Once the tiling of the space is defined, for Lebesgue-almost every line ℓ after finitely many interactions, the Poincaré algorithm fixes one of the vertices of the chain of tetrahedrons, i.e., there exists $k_0 = k(\ell)$ such that

$$A^{(k)} = A^{(k_0)}, \quad \text{for all} \quad k \geq k_0 \text{ ''}.$$

In the same paper [Po2], Poincaré gave a nice and convenient analytical version of his algorithm. Let x_1, x_2, x_3 be the affine coordinates of the point where the line ℓ intersects the triangle ABC (Figure 4). With respect to this triangle assume that $x_1 \geq x_2 \geq x_3$. Let A′B′C′ be the next triangle. In this particular case, A′ = A, B′ = Q_1, C′ = Q_2. Therefore the affine coordinates of this point with respect to the triangle A′B′C′ are

$$x_1 - x_2, \quad x_2 - x_3, \quad x_3.$$

Analytically, the Poincaré algorithm denoted by P corresponds to the following map:

(1.3) $$\mathrm{P}: \quad x = (x_1,\ x_2,\ x_3) \in \mathbb{R}^3_+ \mapsto x' = \left(x_{\sigma(1)}, x_{\sigma(2)} - x_{\sigma(1)}, x_{\sigma(3)} - x_{\sigma(2)}\right),$$

where $\mathbb{R}^3_+$ is the positive cone in $\mathbb{R}^3$ and σ depends on x. Precisely, σ is the permutation which arranges in increasing order the coordinates x_1, x_2, x_3:

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq x_{\sigma(3)}.$$

So, to each point x, P associates a sequence

$$x,\ x^{(1)},\ \ldots,\ x^{(k)},\ \ldots .$$

In this context, Poincaré and Châtelet claimed (see note in [Po2], p. 187) that if the coordinates of x were rationally independent, as k gets large, the coordinates of $x^{(k)}$ become very small. In other words, $x^{(k)}$ converges to 0, as $k \to +\infty$. It will be proved here that a quite different phenomenon occurs in the limit:

"For Lebesgue-almost every point $x \in \mathbb{R}^3_+$, $x^{(k)}$ converges to the point $(0,\ 0,\ r)$, as $k \to +\infty$, where $r > 0$ depends on x."

It is worth mentioning that, independently of Poincaré, recently, number theorists have attempted to prove the convergence of $x^{(k)}$ to 0, as $k \to +\infty$ (Ruzsa [R]).

Poincaré's idea can be extended to higher dimensions. It is natural to call the **Poincaré algorithm**, P, the map defined in $\mathbb{R}^n_+$, for $n \geq 3$, by

$$\text{(1.4)} \quad \mathrm{P}: \quad x = (x_1, \ldots, x_n) \mapsto x' = \left(x_{\sigma(1)}, x_{\sigma(2)} - x_{\sigma(1)}, \ldots, x_{\sigma(n)} - x_{\sigma(n-1)}\right),$$

where σ is the permutation which arranges in increasing order $x_1, \ldots, x_n$. To avoid misunderstanding, the two-dimensional case which reduces to the Euclidean algorithm was called F (1.1).

Section 2 of this paper is dedicated to an interesting rediscovery of the algorithm. Without knowing Poincaré's work, and having a completely different motivation, Daniels [D] and Parry [Pa] considered the same dynamics. Clearly, if one wishes to study the dynamical and metrical properties of an algorithm, it is necessary to normalize it. This should be done in such a manner that the domain remains unchanged. The result of this operation is the so-called Gauss map induced by the algorithm. In [D], Daniels conjectured the ergodicity of the Gauss map T induced by P (1.4),

$$\text{(1.5)} \qquad \mathrm{T}: \quad u \in \Delta_{n-1} \mapsto \frac{Pu}{|Pu|_1} \in \Delta_{n-1},$$

where Δ_{n-1} is the $(n-1)$-dimensional simplex

$$\{(u_1,\ldots,u_n)\in\mathbb{R}^n_+ : u_1+\cdots+u_n=1\}$$

and $|Pu|_1 = u_{\sigma(n)}$.

Geometrically, T maps the point where the line ℓ intersects the triangle ABC into the point where ℓ intersects the triangle A′B′C′, both points given in barycenter coordinates corresponding to the respective triangle.

This paper is concerned with the dynamics of F, P and T. There were attempts to prove Daniels' conjecture, in particular, by Parry [Pa] and Schweiger [S].

Regarding the two-dimensional case, it will be shown in Section 8 that the dynamics of F is equivalent to that of the horocycle flow. From the ergodicity of the former will follow the ergodicity of the latter.

The three-dimensional dynamics have quite remarkable properties. In Section 7, it will be proved that the algorithm P is not ergodic. Nevertheless, the map T is ergodic (Section 8); this follows from the ergodicity of the two-dimensional algorithm F even though T is nonconservative (Section 3), has an attractor (Section 7) and is the projection of a nonergodic map.

Section 4 is devoted to certain continued fraction sums which naturally arise in this article. In Section 5, by completeness, it is shown that Poincaré's idea motivates a geometrical interpretation of multidimensional continued fraction algorithms: for example, Jacobi–Perron, Brun and Selmer algorithms (see Brentjes [B]) and also transformations in the space of interval exchange maps (see Veech [V]). They all bear the same basic idea of approximating a line in the n-dimensional Euclidean space by vertices of $(n+1)$-hedrons whose vertices lie on a fixed lattice.

The proof of existence of an absorbing set for the map T is left to Section 6. Section 9 regards the n-dimensional algorithm and some conjectures are made.

The results proved in this paper lead one to conjecture the following dynamical behaviour for P (1.4) and T (1.5) on dimension n larger than 3:

If n is even: P is ergodic and T is ergodic and conservative.

If n is odd: P is nonergodic and T is ergodic, although nonconservative, and T has an attractor, $T^k x \to (0,\ldots,0,1)$ as $k\to+\infty$, for a.e. x.

2. The work of Kendall, Daniels, Parry and Schweiger

In [Ke], Kendall introduced an empirical probabilistic rank correlation method which is based only upon the measure of certain objects. He was interested in a situation in which one is able not only to rank the original objects in order of magnitude, but also the differences of these magnitudes. Basically, he would then associate, to each object, a finite chain of permutations. His process is defined in the following manner: each point x of the simplex Δ_{n-1} is taken to an element of $S_n \times \cdots \times S_2$. Here S_k denotes the permutation group in k letters:

$$x \mapsto (\sigma_0, \sigma_1, \ldots, \sigma_{n-2}),$$

where σ_0 arranges in increasing order $(x_1, x_2, \ldots, x_n)$ and σ_{k+1} arranges in increasing order

$$\left(x^{(k)}_{\sigma_k(2)} - x^{(k)}_{\sigma_k(1)}, \ldots, x^{(k)}_{\sigma_k(n-k)} - x^{(k)}_{\sigma_k(n-(k+1))}\right) = \left(x_1^{(k+1)}, \ldots, x^{(k+1)}_{n-(k+1))}\right),$$

for $0 \leq k \leq n-3$. To each point $(\sigma_0, \sigma_1, \ldots, \sigma_{n-2})$ corresponds a sub-simplex of Δ_{n-1}. In each sub-simplex, Kendall fixes a point which is chosen by using an averaging method.

He expected that this point would give a good approximation for any other point of the sub-simplex.

Anyway, one of the sub-simplexes has diameter $(1 - \frac{1}{n})^{1/2}$ and therefore this poses a problem in the choice of a "good" approximating point.

The work of Kendall has motivated Daniels [D] to introduce two processes by which a point in the simplex Δ_{n-1} can be expanded as a sequence of permutations. Then, Daniels discussed their behaviour with the assumption of ergodicity. One of these maps is T (1.5), the renormalization of the so-called Poincaré Algorithm. Daniels [D] proved that T admits a σ-finite invariant measure which is absolutely continuous with respect to Lebesgue measure and found its density to be

$$\frac{1}{x_1(x_1+x_2)\cdots(x_1+\cdots+x_{n-1})}.$$

Concerning this map, Parry [P] proved that T is ergodic for $n = 2$ and was unable to decide the case $n > 2$. In [S], Schweiger studies the case $n = 3$, and for this case proves that T is nonconservative, that is, T admits a positive measure wandering set. Throughout this paper $\lambda(.)$, means the Lebesgue measure in Δ_2.

For $n = 3$, the following theorems will be proved:

THEOREM 2.1: *For λ-almost every $x \in \Delta_2$,*

$$\lim_{k \to +\infty} T^k(x) = (0,0,1).$$

In other words, the vertex (0,0,1) is an attractor for T. Theorem 2.1 is equivalent to assertion (1.1) about the Poincaré algorithm.

With respect to the question of ergodicity, one has for the algorithm itself:

THEOREM 2.2: *P is nonergodic.*

Both theorems follow straight from Theorem 5.7, and their proofs are left to Section 7. For the so-called Gauss map T, the next result answers the question initially posed by Daniels [D].

THEOREM 2.3: *T is ergodic (with respect to Lebesgue measure).*

The proof is left to Section 8.

3. The nonconservativeness of T

In order to have a clear picture of what really occurs in the dynamics of P and T, an alternative proof of the Schweiger theorem [S] will be given.

Let $0 < x_1, x_2 < x_3$; if P (1.1) is applied to this triple, one obtains

$$x = (x_1, x_2, x_3) \mapsto$$
$$x' = (\min\{x_1,x_2\},\ \max\{x_1,x_2\} - \min\{x_1,x_2\}, x_3 - \max\{x_1,x_2\}).$$

Assume that x_1 and x_2 are rather smaller than x_3. If one iterates P at this point $k+1$ times, for k not large, one obtains

$$x^{(k+1)} = \Bigg(\min\left\{x_1^{(k)}, x_2^{(k)}\right\}, \max\left\{x_1^{(k)}, x_2^{(k)}\right\} - \min\left\{x_1^{(k)}, x_2^{(k)}\right\}, \quad x_3 - \sum_{i=0}^{k} \max\left\{x_1^{(i)}, x_2^{(i)}\right\} \Bigg), \tag{3.1}$$

The sum which appears in the third coordinate does not depend on x_3 for a suitable k. One sets

$$S(x_1,x_2) = \sum_{k \le 0} \max\{x_1^{(k)}, x_2^{(k)}\}. \tag{3.2}$$

It will be proved that for almost every pair $x_1, x_2 > 0$, the sum (3.2) converges. For these pairs (x_1, x_2), if one chooses $x_3 \geq S(x_1, x_2)$, the point $x = (x_1, x_2, x_3)$ satisfies (3.1) for all k. It implies that

$$x^{(k)} \to (0, 0, x_3 - S(x_1, x_2)), \quad \text{as } k \to +\infty.$$

It is clear that

$$S(x_1, x_2) \text{ converges if, and only if, } S\left(\frac{\min\{x_1, x_2\}}{\max\{x_1, x_2\}}, 1\right) \text{ converges.}$$

As far as the sum S is concerned, it is the rate $\dfrac{\min\{x_1, x_2\}}{\max\{x_1, x_2\}}$ that matters. Set

$$S(\theta) = S(\theta, 1), \quad \text{for } 0 < \theta < 1. \tag{3.3}$$

It will be proved in Lemma 4.1 that for Lebesgue-almost all $0 < \theta < 1$, the sum $S(\theta)$ converges. Define

$$\begin{aligned} \Gamma = &\{(\theta r, r, 1 - (\theta + 1)r), (r, \theta r, 1 - (\theta + 1)r) \colon \theta \in S^{-1}(0, +\infty)\}, \\ &0 \leq r \leq (\theta + 1 + S(\theta))^{-1}. \end{aligned} \tag{3.4}$$

It follows from Lemma 4.1 that $\lambda(\Gamma) > 0$.

Lemma 3.1: $\Gamma = \bigcap_{k \geq 0} \mathbf{T}^{-k} \{x \in \Delta_2 \colon x_1, x_2 \leq x_3\}$.

Proof: Let $x \in \Gamma$. Write $x = (x_1, x_2, 1 - (\theta + 1)r)$. We have that

$$x_1 + x_2 + 1 - (\theta + 1)r = \theta r + r + 1 - (\theta + 1)r = 1,$$

therefore $x \in \Delta_2$.

One has that

$$S(x_1, x_2) = rS(\theta, 1) = rS(\theta) \leq 1 - (\theta + 1)r,$$

by the choice of r. It implies that

$$x \in \bigcap_{k \geq 0} T^{-k} \{x \in \Delta_2 \colon x_1, x_2 \leq x_3\}.$$

Let $x \in \Delta_2$ and $x^{(k)} = T^k x$. Assume that, for all $k \geq 1$,

$$x_1^{(k)}, x_2^{(k)} \leq x_3^{(k)}.$$

Therefore $x_3 - \sum_{k\geq 0} \max\left\{x_1^{(k)}, x_2^{(k)}\right\}$ converges. Thus $x \in \Gamma$. ∎

Note that $T^{-1}(\Gamma)\backslash\Gamma$ is a wandering set, since by definition

$$T(\Gamma) \underset{\neq}{\subset} T \underset{\neq}{\subset} T^{-1}(\Gamma).$$

Therefore T is nonconservative.

4. Continued fraction sums

For the elementary properties of continued fractions used here, see Khintchine [Kh].

Let $0 < \theta_1 < 1$ and consider its continued fraction expansion

$$\theta_1 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \tag{4.1a}$$

The iterates of the Gauss map at the point θ_1 are, for $k \geq 1$,

$$\theta_k = \cfrac{1}{a_k + \cfrac{1}{a_{k+1} + \ddots}}$$

The sum $S(\theta_1)$ is given by the following recursion formula:

$$\begin{aligned} S(\theta_1) &= 1 + (1-\theta_1) + \cdots + (1-(a_1-1)\theta_1) + \theta_1 S(\theta_2) \\ &= a_1\big(1 - \frac{\alpha_1 - 1}{2}\theta_1\big) + \theta_1 S(\theta_2) \end{aligned} \tag{4.2a}$$

which implies that

$$S(\theta_1) = \sum_{k\geq 1} \theta_0\theta_1 \cdots \theta_{k-1} a_k \big(1 - \frac{a_k - 1}{2}\theta_k\big), \tag{4.2b}$$

where $\theta_0 = 1$. Set

$$R(\theta_1) = \sum_{k\geq 1} \theta_0\theta_1 \cdots \theta_{k-1} a_k. \tag{4.3}$$

It follows that

$$\frac{1}{2}R(\theta_1) \leq S(\theta_1) < R(\theta_1). \tag{4.4}$$

Next, it will be proved that S is well defined almost everywhere.

LEMMA 4.1: *The sum $S(\theta)$ converges, for almost every $0 < \theta < 1$.*

Proof: According to (4.4), it is necessary and sufficient to prove that R converges almost everywhere.

For θ_1 (4.1a), let $q_k = q_k(\theta_1)$ be the denominators of its approximants. They are defined by

$$q_{k+1} = a_{k+1}q_k + q_{k-1}, \quad k \geq 0, \quad q_{-1} = 0, \quad q_0 = 1. \tag{4.5}$$

By (4.1) and (4.5), one has

$$\theta_1 = \frac{1}{a_1 + \theta_2} = \frac{1}{q_1 + q_0\theta_2}.$$

Claim that, for $k \geq 1$,

$$\theta_1 \cdots \theta_k = \frac{1}{q_k + q_{k-1}\theta_{k+1}}. \tag{4.6}$$

By (4.1b), $\theta_{k+1} = 1/\left(a_{k+1} + \theta_{k+2}\right)$; one obtains that

$$\begin{aligned}\theta_1 \cdots \theta_k\theta_{k+1} &= \frac{1}{q_k + q_{k-1}\theta_{k+1}}\theta_{k+1} = \frac{1}{q_k\frac{1}{\theta_{k+1}} + q_{k-1}} \\ &= \frac{1}{q_k(a_{k+1} + \theta_{k+2}) + q_{k-1}} = \frac{1}{(a_{k+1}q_k + q_{k-1}) + \theta_{k+2}q_k};\end{aligned}$$

according to (4.5), this proves the claim.

The general term of the sum $R(\theta_1)$ (4.3) satisfies the following inequalities: by (4.6)

$$\frac{a_{k+1}}{2q_k} < \theta_0 \cdots \theta_k a_{k+1} < \frac{a_{k+1}}{q_k}, \tag{4.7}$$

since $q_{k-1}\theta_{k+1} < q_{k-1} < q_k$.

Let $\alpha = \left(\sqrt{5} - 1\right)/2$ be the inverse of the golden ratio; its continued fraction expansion is

$$\alpha = \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}.$$

One concludes that, for all $k \geq 0$,

$$q_k(\alpha) \leq q_k(\theta_1). \tag{4.8}$$

Besides, by (4.1), $\alpha_k = \alpha$ for all k. It implies by (4.7) and (4.8) that

$$\frac{1}{2q_k(\theta_1)} < \alpha^k \quad \text{or} \quad \frac{1}{2}\alpha^{-k} < q_k(\theta_1). \tag{4.9}$$

The Borel–Bernstein Theorem (see [Kh], p. 62) assures that, if $\phi_k > 0$ and $\Sigma(1/\phi_k)$ converges, then, for almost every θ_1, $a_k > \phi_k$ only finitely many times.

Let $\epsilon > 0$; since $\Sigma\alpha^{\epsilon k}$ converges, for almost every θ_1,

$$a_{k+1} < \alpha^{-\epsilon k}, \tag{4.10}$$

for k sufficiently large. From (4.7), (4.9) and (4.10), one obtains

$$\theta_0 \cdots \theta_k a_{k+1} < 2\alpha^k \cdot \alpha^{-\epsilon k} = 2\alpha^{(1-\epsilon)k}, \tag{4.11}$$

for k sufficiently large.

This proves Lemma 4.1. ∎

The sum established in (4.2) can be generalized for higher dimension. Set $S_0(\theta_1) = 1$ and $S_1(\theta_1) = S(\theta_1)$. Now (4.2) reads

$$S_1(\theta_1) = a_1\left(S_0(\theta_1) - \frac{a_1 - 1}{2}\theta_1\right) + \theta_1 S_1(\theta_2),$$
$$S_1(\theta_1) = \sum_{k\geq 1} \theta_0 \cdots \theta_{k-1} a_k\left(S_0(\theta_k) - \frac{a_k - 1}{2}\theta_k\right).$$

Recall that by (4.2),

$$S_1(\theta_1) = 0 + 1 + (1 - \theta_1) + \cdots = \sum_{k\geq 0} C_k^{(1)}, \tag{4.12}$$

where $C_k^{(1)}$ is the k-th term. Set

$$S_2(\theta_1) = 0 + \sum_{k\geq 0}\left(S_1(\theta_1) - \sum_{i=0}^{k} C_i^{(1)}\right) = \sum_{k\geq 0} C_k^{(2)},$$

where $C_0^{(2)} = 0$ and $C_k^{(2)} = S_1(\theta_1) - \sum_{i=0}^{k} C_i^{(1)}$.

By recursion, for $m \geq 1$, define

$$S_{m+1}(\theta_1) = 0 + \sum_{k\geq 0}\left(S_k(\theta_1) - \sum_{i=0}^{k} C_i^{(k)}\right) = \sum_{k\geq 0} C_k^{(m+1)}. \tag{4.13}$$

According to (4.2a)

$$\begin{aligned} S_2(\theta_1) =& S_1(\theta_1) + (S_1(\theta_1) - 1) + \cdots \\ & + (S_1(\theta_1) - (1 + (1 - \theta_1) + \cdots + (1 - (a_1 - 1)\theta_1)) + \cdots \\ & \le a_1 S_1(\theta_1) + \theta_1 S_1(\theta_2) + \cdots \\ =& a_1 S_1(\theta_1) + \theta_1 S_2(\theta_2) \end{aligned}$$

and by recursion, for $m \ge 1$, one obtains

$$S_{m+1}(\theta_1) \le a_1 S_m(\theta_1) + \theta_1 S_m(\theta_2). \tag{4.14}$$

Set

$$\begin{gathered} R_0(\theta_1) = 1, \quad R_1(\theta_1) = R(\theta_1) \ \ (4.3) \quad \text{and} \\ R_{m+1}(\theta_1) = \sum_{k \ge 1} \theta_0 \theta_1 \ldots \theta_{k-1} a_k R_m(\theta_k); \end{gathered} \tag{4.15}$$

therefore, according to (4.14), for $m \ge 0$

$$S_m(\theta_1) \le R_m(\theta_1). \tag{4.16}$$

COROLLARY 4.2: *For almost every $0 < \theta_1 < 1$, all sums $S_m(\theta_1)$ converge, for all $m \ge 0$.*

Proof: According to (4.16), it suffices to prove that for all $m \ge 1$, R_m converges almost everywhere.

By (4.3), for $k \ge 1$,

$$\begin{aligned} R_1(\theta_k) &= a_k + \sum_{l \ge k+1} \theta_k \cdots \theta_{l-1} a_l \\ &= a_k + \frac{1}{\theta_0 \cdots \theta_{k-1}} \sum_{l \ge k+1} \theta_0 \ldots \theta_{l-1} a_l. \end{aligned}$$

According to (4.10) and (4.11), for $k \ge K = K(\theta_1)$,

$$\begin{aligned} R_1(\theta_k) &\le \alpha^{-\epsilon k} + 4\alpha^{k-1} \sum_{j \ge k} \alpha^{(1-\epsilon)j} \\ &= \alpha^{-\epsilon k} \left(1 + 4 \frac{\alpha^{2k-1}}{1 - \alpha^{1-\epsilon}} \right) \\ &\le M \alpha^{-\epsilon k}, \end{aligned}$$

where $M = M(K, \epsilon) > 0$.

By a suitable choice of ϵ, $\epsilon < \frac{1}{2}$, it implies that R_2 (4.15) converges almost everywhere according to (4.11).

Using recursion, it follows that R_m (4.15) converges almost everywhere, for any fixed $m \geq 1$.

This concludes the proof of Corollary 4.2. ∎

5. Partitions

The aim of this section is to define the partitions of the simplex $\Delta = \Delta_2$ which are obtained as T is iterated. This enables one to deal with measurements in the simplex which are necessary in Section 6.

The method used here will be extended to other multidimensional continued fraction algorithms. The geometric interpretation of these algorithms will enable one to introduce some tools which are useful for computation in the simplex. Therefore, one will be able to study metric and ergodic properties of these algorithms.

Recall the Poincaré Algorithm and consider the basic tetrahedron as OABC. Call Δ the triangle ABC. Here the coordinates of the vertices A, B, C are, respectively, (1,0,0), (0,1,0), (0,0,1) since they are the basic points of the lattice defined by the points O,A,B,C. Let $\alpha, \beta > 0$ be real numbers and ℓ be the line: $y = \alpha x, \quad z = \beta x$. Let $(x_1,\ x_2,\ x_3)$ be the barycenter coordinates of the point where ℓ intersects the triangle Δ_2. Let σ be the permutation which arranges in increasing order $x_1,\ x_2,\ x_3$:

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq x_{\sigma(3)}.$$

Let OA′B′C′ be the new tetrahedron defined by P (1.2). Let $M = (m_{ij})$ be the 3×3 matrix defined by

$$m_{ij} = \begin{cases} 0, & \sigma^{-1}(i) < j, \\ 1, & \text{otherwise.} \end{cases}$$

If one writes the coordinates of the points in columns, one obtains

$$Q' = MQ, \quad \text{for} \quad Q = \text{A,B,C}.$$

There are six matrices M_σ each one corresponding to a permutation $\sigma \in S_3$, the group of permutations in three letters. Let

$$L_M: \quad u \in \Delta_2 \mapsto \frac{Mu}{|Mu|_1} \in \Delta_2.$$

The sub-simplex $L_{M_\sigma}(\Delta_2)$ intersects all lines ℓ which intersect the interior of tetrahedron OA′B′C′, for each $\sigma \in S_3$. Set

$$\Delta_\sigma = L_{M_\sigma}(\Delta_2).$$

Therefore

$$\Delta = \bigcup_{\sigma \in S_3} \Delta_\sigma$$

is a disjoint union up to null measure sets. For simplicity, enumerate these six sub-simplexes $\Delta_1', \ldots, \Delta_6'$. Notice that

$$T\Delta_i' = \Delta.$$

Call

$$P_1 = \{\Delta\prime_i\colon 1 \leq i \ \leq 6\}$$

the first partition of Δ. Each sub-simplex Δ_i' is partitioned into six sub-simplexes, as the map P is iterated again. The second partition of Δ is given by

$$P_2 = \{\Delta_1' \cap L_{M_\sigma}(\Delta_i')\colon 1 \leq i \leq 6, \ \ \sigma \in S_3\}.$$

For simplicity, write

$$P_2 = \left\{\Delta_{i_1,\, i_2}''\colon 1 \leq i_1,\ i_2 \leq 6\right\}.$$

Notice that P_2 does not consist of 6^2 equal triangles. By recursion, if

$$P_k = \left\{\Delta_{i_1,\ldots,i_k}^{(k)}\colon 1 \leq i_m \leq 6, \ \ 1 \leq m \leq k\right\} \tag{5.1}$$

is the k-th partition, the $(k+1)$-th is given by

$$P_{k+1} = \left\{\Delta_{i_1,\ldots,i_k}^{(k)} \cap L_{M_\sigma}(\Delta_{i_1,\ldots,i_k}^{(k)})\colon 1 \leq i_m \leq 6,\ 1 \leq m \leq k,\ \sigma \in S_3\right\}.$$

In other words, let $\sigma_i \in S_3$, for $1 \leq i \leq k$, and set

$$M^{(k)} = M_{\sigma_k} \ldots M_{\sigma_1}.$$

One has that P_k is formed by the union of all $L_{M^{(k)}}(\Delta)$.

It was noticed by Veech [V] that if M is a strictly positive $n \times n$ matrix and ℓ_1, ℓ_2, ℓ_3 are the sums of its three columns, the Jacobian of L_M equals

$$J_M(u) = \frac{1}{n!(u_1\ell_1 + \cdots + u_n\ell_n)^n}, \tag{5.2}$$

where u belongs to Δ_{n-1}.

In particular, the area of $L_M(\Delta_{n-1})$ is

$$\lambda(L_M(\Delta)) = \int_0^1 du_1 \cdots \int_0^{1-(u_1+\cdots+u_{n-2})} du_{n-1} J_M(u) = \frac{1}{n!\ell_1\cdots\ell_n}. \tag{5.3}$$

Poincaré's idea of a geometric algorithm can be extended to any multi-dimensional continued fraction algorithm defined in Brentjes ([B], p. 18) and also considered in Arnoux and Nogueira [AN].

In the space $\mathbb{R}^n$, let $A_1, \ldots, A_n$ be the basic points of an n-dimensional lattice. With respect to this lattice, the coordinates of the vertices $A_1, \ldots, A_n$ are, respectively, $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Partition the $(n+1)$-hedron $\mathrm{OA}_1 \ldots \mathrm{A}_n$ into $(n+1)$-hedrons whose vertices belong to the lattice: $\mathrm{O}, \mathrm{A}'_1 \ldots, \mathrm{A}'_n$. By recursion, at the $(k+1)$-th stage the $(n+1)$-hedron $\mathrm{OA}_1^{(k)} \ldots \mathrm{A}_n^{(k)}$ is partitioned into $(n+1)$-hedrons whose vertices belong to the lattice and, moreover, one of them is the origin O. Sometimes this partition is infinite, as is the case of the Jacobi–Perron algorithm.

Let $\alpha_1, \ldots, \alpha_{n-1} > 0$ be real numbers and assume that the line ℓ: $x_2 = \alpha_1 x_1, \ldots, x_n = \alpha_{n-1} x_1$, intersects the interior of the $(n+1)$-hedron $\mathrm{OA}_1 \ldots \mathrm{A}_n$. One defines the algorithm which associates to the line ℓ the ordered sequence of $(n+1)$-hedrons which contain ℓ in their interiors:

$$\mathrm{OA}_1 \cdots \mathrm{A}_n \to \mathrm{OA}'_1 \cdots \mathrm{A}'_n \to \mathrm{OA}_1^{(k)} \cdots \mathrm{A}_n^{(k)} \to \cdots.$$

Analytically, this algorithm is defined in the following manner. Let Δ_{n-1} be the $(n-1)$-dimensional simplex $\mathrm{A}_1 \ldots \mathrm{A}_n$. Let $x = (x_1, \ldots, x_n)$ be the barycenter coordinates of the point where the line ℓ intersects the simplex. The barycenter coordinates $(x_1^{(k)}, \ldots, x_n^{(k)})$ of the point x with respect to the $(n-1)$-hedron $\mathrm{A}_1^{(k)} \ldots \mathrm{A}_n^{(k)}$ are the k-th iterate of the algorithm at point x.

Write all vertices in columns. There exists a matrix M with positive integer coefficients which gives

$$A_i^{(k)} = MA_i, \quad \text{for } 1 \le i \le n.$$

It implies that $\det M = \pm 1$.

As before, one can define the sub-simplex associated to the matrix M:

$$L_M(\Delta_{n-1}). \tag{5.4}$$

The partition P_k defined by the algorithm into Δ_{n-1} is the disjoint union of the sub-simplexes (5.4), when M runs over all suitable matrices.

6. An absorbing set for T

The aim of this section is to prove that the set

$$\Gamma_\infty = \bigcup_{k\geq 0} T^{-k}\Gamma \tag{6.1}$$

equals Δ_2 almost surely. This means:

THEOREM 6.1: *Γ is an absorbing set.*

Let M be a strictly positive 3×3 matrix corresponding to one of the triangles defined in (5.1):

$$\Delta_i^{(k)} = L_M(\Delta_2). \tag{6.2}$$

It follows from (5.2) that

$$\lambda(L_M(\Delta_2)\cap \bigcup_{0\leq i\leq k} T^{-i}\Gamma) = \int_\Gamma J_M.$$

Let ℓ_1, ℓ_2, ℓ_3 be the sums of the columns of M. If one introduces a suitable change of variables (3.4), one obtains

$$\begin{aligned}&\lambda(L_M(\Delta_2)\cap T^{-k}\Gamma)\\ &= \frac{1}{2}\sum_{\sigma\in S_2}\int_0^1 d\theta \int_0^{(\theta+1+S(\theta))^{-1}} dr \frac{1}{(\ell_{\sigma(1)}\theta+\ell_{\sigma(2)}+\ell_3(1-\theta r - r))^3}\\ &= \frac{1}{2}\sum_{\sigma\in S_2}\frac{1}{\ell_3}\int_0^1 \frac{d\theta}{(\ell_{\sigma(1)}\theta+\ell_{\sigma(2)}+\ell_3 S(\theta))^2}.\end{aligned} \tag{6.3}$$

According to (5.3) and (6.3), the following has been proved:

LEMMA 6.1: *Let $x\in\Delta_i^{(k)}$ (6.2). The probability $p(M)$ that $x\in\Gamma_\infty$ (6.1) is greater than*

$$\sum_{\sigma\in S_2}\ell_{\sigma(1)}\ell_{\sigma(2)}\int_0^1 \frac{d\theta}{(\ell_{\sigma(1)}\theta+\ell_{\sigma(2)}+\ell_3 S(\theta))^2}.$$

Here, the probability means the ratio

$$p(M) = \frac{\lambda(\Delta_i^{(k)}\cap\Gamma_\infty)}{\lambda(\Delta_i^{(k)})}. \tag{6.4}$$

From Lemma 4.1, it follows that there exists a constant $K > 0$ such that

$$\lambda(S^{-1}(O,K)) > \frac{1}{2}. \tag{6.5}$$

One concludes, from (6.4) and (6.5), that

$$\begin{aligned} p(M) &\geq \sum_{\sigma\in S_2} \ell_{\sigma(1)}\ell_{\sigma(2)} \int_{1/2}^{1} \frac{d\theta}{(\ell_{\sigma(1)}\theta + \ell_{\sigma(2)} + \ell_3 K)^2} \\ &= \frac{1}{2}\sum_{\sigma\in S_2} \frac{\ell_{\sigma(1)}\ell_{\sigma(2)}}{(\ell_{\sigma(1)} + \ell_{\sigma(2)} + K\ell_3)^2}. \end{aligned} \tag{6.6}$$

Recall that K does not depend on $\ell_1,\ \ell_2, \ell_3$.

From (6.6), one concludes the following:

LEMMA 6.2: *Let M be a strictly positive matrix such that $l < \ell_3 \leq \ell_2 \leq \ell_1 \leq 2\ell_2$, then the probability* (6.4)

$$p(M) \geq \frac{1}{2(K+3)^2}.$$

The next result establishes the central property of the partitions P_k (5.1), $k \geq 1$. If Γ were not absorbant,

$$\lambda(\Gamma_\infty^c) > 0,$$

where $\Gamma_\infty^c = \Delta_2 \backslash \Gamma_\infty$.

LEMMA 6.3: *If $\lambda(\Gamma_\infty^c) > 0$, given $\epsilon > 0$ there exists a sub-simplex $\Delta' \in \bigcup_{k\geq 0} P_k$ such that*

$$\lambda(\Delta' \cap \Gamma_\infty) < \epsilon\lambda(\Delta').$$

Proof: Let $\epsilon > 0$ be fixed. By the Lebesgue Density Theorem (see Fernandez [Fe], pp. 163–164), there exists an open ball of positive radius B contained in Δ_2 such that

$$\lambda(B \cap \Gamma_\infty) < \epsilon\lambda(B).$$

Let x be a density point of $B \cap \Gamma_\infty^c$.

First it is claimed that there exists a sub-simplex $\Delta(x)$ which satisfies the following:

$$\Delta(x) \in \bigcup_{k\geq 0} P_k, \tag{6.7a}$$

$$\Delta(x) \subset B, \tag{6.7b}$$

$$x \in \Delta(x). \tag{6.7c}$$

Since x is not in Γ_∞, x and its iterates have no isolated coordinate. For all $k \geq 1$, let

$$x \in \Delta_{i_k} = L_{M^{(k)}}(\Delta_2), \quad \text{where } 1 \leq i_k \leq 6.$$

Let

$$\ell^{(k)} = \begin{pmatrix} \ell_1^{(k)} \\ \ell_2^{(k)} \\ \ell_3 \end{pmatrix} = {}^tM^{(k)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{6.8}$$

be the column sums of the matrix $M^{(k)}$.

The partitions P_k define the following map:

$$x \mapsto (\sigma_k)_{k\geq 0} \in S_3^N. \tag{6.9}$$

If $x \in \Gamma_\infty^c$ infinitely often

$$\sigma_k(3) < 3.$$

In the three-dimensional case, the above condition implies that the enclosed sub-simplexes $\Delta_k(x) \in \bigcup P_m$ which contain x shrink to x as $k \to +\infty$. The matrices $M^{(k)}$ have no isolated column: to each column are added the two other ones infinitely often. This proves the claim.

Let $k(x)$ be the minimum of $k \geq 0$ such that P_k contains a sub-simplex which satisfies (6.7). Note that if P_k satisfies this condition, P_{k+1} also does.

Set, for a fixed k,

$$V_k = \bigcup \{\Delta' \in P_k : \ \Delta' \ \text{ satisfies (6.7) for some density point of } B \cap \Gamma_\infty^c\};$$

thus

$$V_k \subset V_{k+1} \quad \text{and}$$

$$\lambda(V_k) \to \lambda(B \cap \Gamma_\infty^c), \quad \text{as } k \to +\infty.$$

This implies that there exists k such that

$$\lambda(V_k) > \frac{1}{2}\lambda(B \cap \Gamma_\infty^c),$$

therefore there exists a triangle $\Delta' \in P$ such that

$$\lambda(\Delta' \cap \Gamma_\infty) < 2\epsilon\lambda(\Delta').$$

This proves the assertion. ∎

Let $x \in \Delta_2$ and $\{\sigma_k\}_{k\geq 0}$ be the sequence given by (6.9). Recall that σ_k arranges in ascending order $x^{(k)} = T^k x$.

For simplicity, impose that for $a < b$ the permutation which arranges in increasing order: (a,a,a), (a,a,b) and (a,b,b) is (1).

(b,b,a)	is (123).
(a,b,a)	is (23).
(b,a,b)	is (12).
(b,a,a)	is (132).

LEMMA 6.4: *Let $x \in \Delta_2$ and $x_1 > 0$, then there exists $k \geq 0$ such that*

$$\sigma_k(1) > 1.$$

Proof: Consider such an x. There exists an integer $m > 1$ for which $mx_1 > \min\{x_2,\ x_3\}$. Therefore $x_1 > \min\{x_2^{(m)}, x_3^{(m)}\}$ implies that for some $0 \leq k \leq m$,

$$\sigma_k(1) > 1. \qquad \blacksquare$$

Let $x \in \Delta_2$ with $x_1 > 0$; set

$$k(x) = \min\{k \geq 0\colon \sigma_k(1) > 1\}. \tag{6.10}$$

According to Lemma 6.4, $k(x)$ is finite.

The next result will allow one to apply Lemma 6.2.

LEMMA 6.5: *Let $x \in \Delta_2$, with $x_1 > 0$; then there exists k for which $\ell^{(k)}$ (6.8) satisfies the following:*

$$\frac{1}{2} \leq \frac{l^{(k)}_{\sigma_k(1)}}{l^{(k)}_{\sigma_k(2)}} \leq 2, \tag{6.11}$$

Proof: For ease, assume $0 < x_1 \leq x_2 \leq x_3$. If $l = l^{(k)}$ satisfies the desired condition (6.11), one is done. If not, but $\max\{l_2, l_3\} \geq l_1$, then l' satisfies (6.11). So one is left with the case where $l_1 > \max\{l_2, l_3\}$. Here at time $k = k(x)$ (6.10), $\max\left\{l_2^{(k)},\ l_3^{(k)}\right\} \geq l_1^{(k)}$, and therefore $l^{(k+1)}$ satisfies condition (6.11). $\blacksquare$

Now Theorem 6.1 will be proved.

For $l \in \mathbb{R}^3_+$ and $x \in \Delta_2$, with $x_1 > 0$, according to Lemma 6.5

$$t(x) = \min\{k \geq 0\colon l^{(k)} \text{ satisfies (6.11)}\}$$

is finite. It implies that, for any fixed initial l,

$$\Delta_2 = \bigcup_{t \geq 0} \{x : t(x) = t\} \quad \text{almost surely,}$$

where the right-hand side is a disjoint union. Therefore, given a $\delta > 0$, there exists a $t_0 = t(\delta)$ for which

$$\delta\{x\colon t(x) \geq t_0\} < \delta. \tag{6.12}$$

Let Δ' be a sub-simplex which is given by Lemma 6.3 for

$$\epsilon = \frac{1}{4(K+3)^2}, \tag{6.13}$$

where the constant K satisfies (6.5).

Applied to Δ', Lemma 6.5 and (6.12) imply that there exists a $t_0 = t(\delta)$ such that

$$\lambda\{x \in \Delta'\colon t(x) \geq t_0\} < \delta\lambda(\Delta'). \tag{6.14}$$

Lemma 6.2 assures that

$$\lambda\{x \in \Delta' \cap \Gamma\colon t(x) \leq t_0\} \geq \frac{1}{2(K+3)^2}\lambda\{x \in \Delta'\colon t(x) \leq t_0\}. \tag{6.15}$$

According to (6.14) and (6.15), one concludes that

$$\lambda(\Delta' \cap \Gamma_\infty) \geq \frac{1-\delta}{2(K+3)^2}\lambda(\Delta'). \tag{6.16}$$

For $\delta \in (0, \frac{1}{2})$, (6.16) contradicts the choice of Δ' given by (6.13).

Theorem 6.1 is proved. ∎

7. Existence of an attractor for T and nonergodicity of P

First Theorem 2.1 will be proved. Let $x \in \Gamma$. For P (1.2), set

$$P^k x = x^{(k)}$$

where $x_3^{(k)} = x_3 - \sum_{i=1}^k \left(x_1^{(i)} + x_2^{(i)}\right)$. In this case, on the two first coordinates P behaves as the Euclidean algorithm (1.4)

$$\left(x_1^{(k)}, x_2^{(k)}\right) \to (0,0), \quad \text{as } k \to +\infty,$$

and

$$x_3^{(k)} \to x_3 - S(x_1, x_2), \quad \text{as} \ \ k \to +\infty,$$

where S is defined in (3.2).

By the definition of T, one obtains

$$T^{k+1}x = \frac{1}{x_3^{(k)}} x^{(k+1)} \to (0,0,1), \quad \text{as} \ \ k \to +\infty. \tag{7.1}$$

By Theorem 6.1, for almost every $x \in \Delta_2$ there exists $m = m(x) \geq 0$ such that $T^m x \in \Gamma$. This implies by (7.1) that

$$T^k x \to (0,0,1), \quad \text{as} \ \ k \to +\infty,$$

for almost every $x \in \Delta_2$.

This shows that (0,0,1) is almost everywhere an attractor point for T.

This concludes the proof of Theorem 2.1. ∎

Next Theorem 2.2 will be proved.

In order to prove that P is not ergodic, we will exhibit a nonconstant function which is invariant by P. Here Γ and Γ (3.4) correspond to the same sets as P replaces T.

Let $x \in \Gamma$; define

$$f(x) = x_3 - S(x_1, x_2), \tag{7.2}$$

where $x = (x_1, x_2, x_3)$ and S is given by (3.2). Note that f is finite and well defined in all Γ. Let $x \in \Gamma_\infty$ (6.1). Set

$$m(x) = \min\{m \geq 0 \ \text{ and } P^m x \in \Gamma\}$$

and

$$f(x) = f(P^{m(x)}(x)). \tag{7.3}$$

According to Theorem 6.1, this defines a positive-valued function in almost all Δ_2.

Claim that $f \circ P = f$.

Let $x \in \Gamma$; from (7.2),

$$f(Px) = x_3' - S(x_1', x_2') = (x_3 - (x_1' + x_2')) - (S(x_1, x_2) - (x_1' + x_2')) = f(x). \tag{7.4}$$

Let $x \in \Gamma_\infty \backslash \Gamma$; by (7.3),

$$f(Px) = f(P^{m(Px)}(Px)),$$

where $m(Px) = m(x) - 1$, since $x \in \Gamma_\infty \backslash \Gamma$. Therefore, by (7.3),

$$f(Px) = f(P^{m(x)}x) = f(x). \tag{7.5}$$

The equalities (7.4) and (7.5) assert the nonergodicity of P and conclude the proof of Theorem 2.2. ∎

8. The Euclidean algorithm and the ergodicity of T

For the dynamical properties of unimodular flows used here, the reader is referred to Furstenberg [Fu] and Ghys [G].

Recall that F (1.1) is defined through two elementary matrices:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}. \tag{8.1}$$

Let s be an irrational number and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

Set

$$t = \frac{as + b}{cs + d};$$

then the continued fraction expansions of s and t have the same tail, that is,

$$s = [a_0, a_1, \ldots, a_m, c_0, c_1, \ldots] \quad \text{and} \quad t = [b_0, b_1, \ldots, b_n, c_0, c_1, \ldots] \tag{8.2}$$

(see Hardy and Wright [HW], Theor. 175, p. 142). Moreover, the converse holds.

Let $x = (x_1, x_2) \in \mathbb{R}^2_+$ be a point with an irrational ratio x_1/x_2. Assume $Ax \in \mathbb{R}^2_+$, for some $A \in \mathrm{GL}(2, \mathbb{Z})$. According to (8.2), there exist positive integers k and l such that $F^k(x)$ and $F^l(Ax)$ are parallel. There are products B and C of elementary matrices (8.1) which allow one to write

$$F^k(x) = Bx \quad \text{and} \quad F^l(Ax) = CAx. \tag{8.3}$$

Therefore x and $B^{-1}CAx$ are parallel. If x is an eigenvector of the matrix $B^{-1}CA$, the ratio of x must be a quadratic surd. Otherwise $B^{-1}CA$ is the identity, that is,

$$F^k(x) = F^l(Ax).$$

Assume the ratio of x is a quadratic surd. There exist positive integers k and l such that Bx and CAx (see (8.3)) have a ratio with periodic continued fraction expansion. Let m be the smallest strict positive integer such that

$$Bx \quad \text{and} \quad F^m(Bx) = DBx$$

are parallel. The direction of Bx is also invariant under CAB^{-1}. It follows that for some integer n

$$CAB^{-1} = D^n.$$

This result, for instance, can be found in Hurwitz and Kritikos [HK] (Proposition 1, p. 246).

The following has been proved:

PROPOSITION 8.1: *Let $x \in \mathbb{R}^2_+$ be a point with an irrational ratio and $Ax \in \mathbb{R}^2_+$, for some $A \in \mathrm{GL}(2,\mathbb{Z})$. Then, for some positive integers k and l,*

$$F^k(x) = F^l(Ax).$$

Proposition 8.1 assures that the action of the group $\mathrm{GL}(2,\mathbb{Z})$ restricted to the positive cone $\mathbb{R}^2_+$ is orbit equivalent to F. In other words, let R_1 and R_2 be equivalence relations on $\mathbb{R}^2_+$ given by

$$\begin{aligned} &xR_1y \Leftrightarrow y = Ax \quad \text{for some} \quad A \in \mathrm{GL}(2,\mathbb{Z}), \\ &xR_2y \Leftrightarrow F^k(x) = F^l(y) \quad \text{for some positive integers } k \text{ and } l. \end{aligned} \tag{8.4}$$

So, R_1 and R_2 are equal.

The ergodicity of the so-called horocycle flow was established by Hedlund [H]: It means that the left shift $\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)$ acts ergodically on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.

This can be rephrased in two manners: any nonvanishing subset of $\mathrm{SL}(2,\mathbb{R})$ invariant to the left under the shift $\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)$ and to the right under the action of $\mathrm{SL}(2,\mathbb{Z})$ is a full measure set. Or, otherwise, since $\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\backslash\mathrm{SL}(2,\mathbb{R})$ is isomorphic to $\mathbb{R}^2 - \{0\}$, any nonvanishing subset of $\mathbb{R}^2\backslash\{0\}$ invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ is a full measure set. The last implies that the action of the group

GL$(2,\mathbb{Z})$ on the plane $\mathbb{R}^2$ is ergodic. Also, the equivalence relation R_1 (8.4) on the positive cone $\mathbb{R}^2_+$ is ergodic.

One summarizes:

COROLLARY 8.2: *The Euclidean algorithm F is ergodic (with respect to Lebesgue measure).*

Next, a description of the dynamics of the three-dimensional algorithm P will be given.

The same notation as in (3.4) and Lemma 3.1 is purposely used to define

$$\Gamma = \bigcap_{k\geq 0} P^{-k}\{x \in \mathbb{R}^3_+ : x_1, x_2 \leq x_3\}. \tag{8.5}$$

By Lemma 4.1, the map

$$\Psi: x = (x_1, x_2, x_3) \in \Gamma \mapsto (x_1, x_2, f(x)) \in \mathbb{R}^2_+ \times \mathbb{R}_+$$

is well defined up to a null measure set, where f is defined in (7.2). By definition, Ψ is injective. Let $x \in \Gamma$ and $c \geq 0$; according to (7.2), one has that

$$\Psi(x_1, x_2, c + S(x_1, x_2)) = (x_1, x_2, c).$$

This proves that Ψ is onto and measure preserving. Thus, Ψ is a measure preserving isomorphism.

The dynamics of P on Γ (8.5) is described in the cartesian product $\mathbb{R}^2_+ \times \mathbb{R}_+$ by

$$F \times \mathrm{id} = \Psi \circ P \circ \Psi^{-1}.$$

Corollary 8.2 implies that any invariant function under $F \times \mathrm{id}$ depends almost everywhere (a.e.) only on the third coordinate.

This proves that any function invariant under the restriction of P to Γ (8.5) is of the form $h \circ f$, for some measurable function $h : \mathbb{R}_+ \to \mathbb{R}$. By (7.3), any function invariant under P is uniquely defined a.e. by its restriction to Γ:

If $x \in \mathbb{R}^3_+$, then $P^{n(x)}(x) \in \Gamma$.

$g \circ P = g$ implies that $g(x) = g(P^{n(x)}(x))$.

About the ergodic decomposition of P, one has established that:

THEOREM 8.3: *The function f (7.3) generates the algebra of invariant functions under P. That is, any invariant function under P is of the form $h \circ f$, for some function $h\colon \mathbb{R}_+ \to \mathbb{R}$.*

Let g be an invariant function under T (1.5); then

$$x \in \mathbb{R}^3_+ \mapsto g\left(\frac{x}{|x|_1}\right)$$

is an invariant function under P. According to Theorem 8.3, there exists h such that

$$g\left(\frac{x}{|x|_1}\right) = h(f(x)), \quad \text{for all } x.$$

By (7.3), $f(tx) = tf(x)$, for any $t > 0$. One obtains that

$$g\left(\frac{x}{|x|_1}\right) = h(tf(x)), \quad \text{for any } t > 0.$$

It follows that h is constant almost everywhere.

Theorem 2.3 has been proved. ∎

9. On higher dimensions ...

Consider the iterates of P (1.3) at $x \in \mathbb{R}^n_+$, for all $k \geq 0$,

$$P^k x = \left(x_1^{(k)}, \ldots, x_n^{(k)}\right). \tag{9.1}$$

Let σ be the permutation which arranges, in increasing order, the coordinates in (9.1) and set, for $1 \leq i \leq n$,

$$y_i^{(k)} = x_{\sigma(i)}^{(k)} \quad \text{and} \quad y_0^{(k)} = 0.$$

LEMMA 9.1: *For each $1 \leq i \leq n$ fixed, the sequence $(y_i^{(k)})_{k\geq 0}$ is decreasing.*

Proof: For $1 \leq i \leq n$, one of the following happens:

$$y_i^{(k+1)} \leq y_i^{(k)} - y_{i-1}^{(k)} \quad \text{or} \quad y_i^{(k)} - y_{i-1}^{(k)} < y_i^{(k+1)}.$$

The last inequality implies that

$$y_i^{(k+1)} \leq \max\left\{y_j^{(k)} - y_j^{(k)} \colon 1 \leq j < i\right\} \leq y_i^{(k)}.$$

This proves the lemma. ∎

Lemma 9.1 allows one to define, for $0 \leq i \leq n$,

$$y_i^{(\infty)} = \lim_{k\to+\infty} y_i^{(k)}.$$

LEMMA 9.2: *For every $x \in \mathbb{R}_+^n$ and all $1 \le i < n$,*

$$y_i^{(\infty)} = 0.$$

Proof: Assume that there exists $1 \le j < n$ such that $y_j^{(\infty)} > 0$ and $y_{j-1}^{(\infty)} = 0$. It implies that for k sufficiently large

$$y_{j-1}^{(k)} < y_j^{(\infty)} \quad \text{and} \quad y_{j+1}^{(k)} < y_{j+1}^{(\infty)} + y_j^{(\infty)}. \tag{9.2}$$

First assume that $y_j^{(\infty)} < y_{j+1}^{(\infty)}$. There exists k sufficiently large for which (9.2) holds and, besides,

$$y_j^{(k)} < y_{j+1}^{(\infty)}. \tag{9.3}$$

(9.2) and (9.3) imply that

$$y_{j+1}^{(k+1)} = \max\left\{y_i^{(k)} - y_{i-1}^{(k)} \colon 1 \le i \le j+1\right\} < y_{j+1}^{(\infty)}.$$

This contradicts Lemma 9.1.

Therefore $y_j^{(\infty)} = y_{j+1}^{(\infty)}$. (8.2) gives that

$$\max\{y_i^{(k)} - y_{i-1}^{(k)} \colon i = 1, \cdots, j-1, j+1\} < y_j^{(\infty)}. \tag{9.4}$$

The left-hand side of (9.4) equals $y_j^{(k+1)}$ *or* $y_{j+1}^{(k+1)}$, so it contradicts Lemma 9.1.

One concludes that $y_i^{(\infty)} = 0$, for all $1 \le i < n$, which proves the lemma. ∎

In the case $n = 4$, Theorem 2.1 allows one to prove the following:

LEMMA 9.3: *Let P be the four-dimensional Poincaré algorithm; then for almost all x*

$$y_4^{(\infty)} = 0.$$

Proof: Set

$$\Lambda = \left\{x \in \mathbb{R}_+^4 \colon y_4^{(\infty)} > 0\right\}. \tag{9.5}$$

Let $x \in \Lambda$. By Lemma 9.2, $y_3^{(\infty)} = 0$, and this allows one to define

$$k(x) = \min\{k \colon y_3^{(k)} \le y_4^{(\infty)}\}.$$

This assures the following partition of Λ:

$$\Lambda = \bigcup_{k\geq 0} \{x \in \mathbb{R}^4_+ : k(x) = k\}.$$

For a fixed k, set

$$\Lambda' = P^k\{x: k(x) = k\}. \tag{9.6}$$

If $x \in \Lambda'$, for all $m \geq 0$,

$$P^{m+1}x = \left(U(x_1^{(m)}, x_2^{(m)}, x_3^{(m)}), x_4^{(m)} - \max\left\{x_1^{(m)}, x_2^{(m)}, x_3^{(m)}\right\}\right), \tag{9.7}$$

where U is the three-dimensional Poincaré algorithm.

Theorem 2.1 assures that for almost all (x_1, x_2, x_3),

$$U^m(x_1, x_2, x_3) \to (0, 0, r), \quad \text{as } m \to +\infty, \tag{9.8}$$

where $r > 0$. (9.7) and (9.8) imply that Λ' (9.6) is a null measure set, and one concludes that Λ (9.5) must also be a null measure set. ∎

One can add to the conjectures made in the introduction of this paper the following:

If n is even: $\sum_{k\geq 0} y_n^{(k)}$ converges for a.e. x.

If n is odd: $y_n^{(\infty)} > 0$ for a.e. x.

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